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Spinning Null Fluid in General Relativity

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Abstract

Exterior and interior solutions of Einstein's equations are given for fluid moving with the speed of light and having a superposed spin. The spin is microscopic and does not refer to the rotation of world lines, which are straight. A strange feature is that the exterior solution is in every case locally isometric to an exterior solution for a non-spinning null fluid.

1. *Introduction*

In previous papers (Bonnor, 1969, 1970) I have described solutions of Einstein's equations in which the sources move in a straight line with the speed of light, c. In one case the source is a null fluid, and in the second a charged null fluid. In this paper I shall give solutions in which the source is a spinning null fluid. Particles made out of these three continua will be called a *nullicon*, a *charged nullicon* and a *spinning nullicon*, respectively.

The field equations used are

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -8\pi T_{ik} \tag{1.1}$$

and the solutions are global ones, consisting of an exterior part for which $T_{ik} = 0$, and an interior part for which $T_{ik} \neq 0$, appropriate conditions being satisfied at the boundary between the two regions.

In Section 2 I present the metric and its field equations, and in Section 3 I obtain the exterior solutions. I show in Section 4 how these can be extended to give globally regular solutions. The physical interpretation follows in Section 5, and here the justification for referring to *spinning* null fluid is given. An exact solution for a pulse of spinning null fluid is given in Section 6, and the superposition property of the solutions is referred to in Section 7.

2. *The Metric and its Field Equations*

A sufficiently general metric is

$$ds^2 = -dx^2 - dy^2 + du(2\alpha\, dx + 2\beta\, dy + 2\, dv + 2A\, du) \tag{2.1}$$

where $\alpha(x,y,u)$, $\beta(x,y,u)$ and $A(x,y,u)$ are functions of class C^1, piecewise C^3, and $-\infty < x,y,u,v < \infty$. It will be assumed throughout that all other functions introduced have sufficient differentiability to fulfil these requirements on α, β and A. Metrics of this type have been studied previously (Geroch, 1966; Wyman & Trollope, 1965). The coordinates will be numbered

$$x^1 \equiv x, \qquad x^2 \equiv y, \qquad x^3 \equiv v, \qquad x^4 \equiv u \tag{2.2}$$

so that x^1 and x^2 are space-like, and x^3 null and x^4 are time-like. The metric has in general one Killing vector (Ehlers and Kundt, 1962)

$$s^i = \sqrt{(2)}\,\delta_3{}^i \tag{2.3}$$

which is null and normal to the null coordinate hypersurfaces

$$x^4 = u = \text{const.} \tag{2.4}$$

The metric (2.1) has $|g_{ik}| = -1$ and

$$g^{ik} = \begin{pmatrix} -1 & 0 & \alpha & 0 \\ 0 & -1 & \beta & 0 \\ \alpha & \beta & -(\alpha^2+\beta^2+2A) & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.5}$$

The transformation

$$v = v^* + \mu(x,y,u) \tag{2.6}$$

leaves the form of the metric (2.1) unaltered but takes α, β and A into

$$\alpha^* = \alpha + \mu_1, \qquad \beta^* = \beta + \mu_2, \qquad A^* = A + \mu_4 \tag{2.7}$$

where $\mu_i \equiv \partial u/\partial x^i$. Hence, if μ is chosen to satisfy

$$\mu_{11} + \mu_{22} = -(\alpha_1 + \beta_2) \tag{2.8}$$

we have

$$\alpha_1{}^* + \beta_2{}^* = 0 \tag{2.9}$$

Hence, without loss of generality we may take

$$\alpha_1 + \beta_2 = 0 \tag{2.10}$$

in (2.1), *and this will be done henceforth.*

It follows from (2.10) that there exists a function ψ such that

$$\alpha = \frac{\partial\psi}{\partial y}, \qquad \beta = -\frac{\partial\psi}{\partial x} \tag{2.11}$$

Let us define w by

$$w = \alpha_2 - \beta_1 = \nabla^2\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \tag{2.12}$$

The only non-zero components of the covariant Ricci tensor are

$$R_{14} = R_{41} = -\tfrac{1}{2}w_2 \tag{2.13}$$

$$R_{24} = R_{42} = \tfrac{1}{2}w_1 \tag{2.14}$$

$$R_{44} = -\nabla^2 A - \tfrac{1}{2}w^2 \tag{2.15}$$

(2.10) and (2.12) having been used. Notice that the scalar curvature R vanishes; hence the *field equations* (1.1) *may be written*

$$w_2 = +16\pi T_{14} = +16\pi T_{41} \tag{2.16}$$

$$w_1 = -16\pi T_{24} = -16\pi T_{42} \tag{2.17}$$

$$\nabla^2 A + \tfrac{1}{2}w^2 = +8\pi T_{44} \tag{2.18}$$

3. *Empty Space Solutions*

In empty space T_{ik} vanishes, so from (2.16)–(2.18) we have

$$\nabla^2 A + \tfrac{1}{2}w^2 = 0, \qquad w = h(u) \tag{3.1}$$

h being arbitrary. Equations (2.11) and (2.12) then imply that α and/or β will contain terms linear in x and/or y. This I shall exclude because I shall confine this work to fields with sources in only the finite part of every 2-surface $u = \text{const.}$, $v = \text{const.}$ *Hence in empty space let us take*

$$\nabla^2 A = 0, \qquad w = \nabla^2 \psi = 0 \tag{3.2}$$

From (2.11) and (3.2) it follows that α and β are conjugate harmonic functions; in particular

$$\alpha_2 = \beta_1 \tag{3.3}$$

so there exists a harmonic function $\phi(x, y, u)$ such that

$$\alpha = \frac{\partial \phi}{\partial x}, \qquad \beta = \frac{\partial \phi}{\partial y} \tag{3.4}$$

Because of this the terms $\alpha dx + \beta dy$ in (2.1) form at fixed u a perfect differential, and by using a transformation of form (2.6) with $\mu = -\phi$ (2.1) can be reduced to the plane-fronted wave metric of form

$$ds^2 = -dx^2 - dy^2 + 2\,du\,dv^* + 2\left(A - \frac{\partial \phi}{\partial u}\right)du^2 \tag{3.5}$$

where

$$\left(A - \frac{\partial \phi}{\partial u}\right)$$

is harmonic in x and y. However, it may not be possible to retain Euclidean topology for the 2-surfaces $u = \text{const.}$, $v = \text{const.}$, and keep

$$\left(A - \frac{\partial \phi}{\partial u}\right)$$

continuous (see Section 6).

To sum up, *we obtain a solution for empty space if the functions* α, β *and* A *in* (2.1) *are such that* (2.11) *and* (3.2) *are satisfied.* Although in empty space α and β can be removed by a coordinate transformation, I prefer not to do this, for reasons which will become clear in Section 6.

4. *Global Solutions*

To generate an exterior solution, ψ and A must satisfy (3.2). Except in the uninteresting case in which they are functions of u only, ψ and A cannot be bounded for all x and y. Let us take the case in which the singularities in ψ and A all lie inside a region D bounded by a cylinder Σ such that

$$\Sigma\colon x^2 + y^2 = \beta^2 \qquad (\text{const.} > 0) \tag{4.1}$$

Inside Σ, replace ψ by some class C^4 function $\psi^*(x,y,u)$ and A by some C^3 function $A^*(x,y,u)$ such that on Σ itself

$$\begin{aligned} &A = A^*, \qquad \frac{\partial A}{\partial x^i} = \frac{\partial A^*}{\partial x^i} \\ &\psi = \psi^*, \qquad \frac{\partial \psi}{\partial x^i} = \frac{\partial \psi^*}{\partial x^i}, \qquad \frac{\partial^2 \psi}{\partial x^{i2}} = \frac{\partial^2 \psi^*}{\partial x^{i2}} \end{aligned} \tag{4.2}$$

It is fairly obvious that the functions ψ^* and A^* exist: I have previously given a proof in the case where $\psi = \psi^* = 0$ (Bonnor, 1969).

Now, ψ^* and A^* will not satisfy (3.2) throughout D; where they do not, it follows from (2.16)–(2.18) that $T_{ik} \neq 0$, so that spinning null fluid is present in D (justification in Section 5). In this way *we can construct globally regular solutions representing the field of spinning null fluid within* Σ.

In the foregoing we generated the global solution by starting with the empty space solutions ψ and A. An alternative procedure, closely analogous to that of classical potential theory, is to prescribe w and T^{33} within Σ (proportional respectively to the angular momentum density and the energy density), and to solve the Poisson equations (2.12) and (2.18) for w and A. If the functions w and T^{33} are sufficiently smooth, the regularity conditions on α, β and A will be satisfied.

If ψ and A are required to vanish at $x, y = \infty$ the solutions of (2.12) and (2.18) for ψ and A are unique, which leads to uniqueness in the g_{ik}. I shall actually relax the conditions imposed at $x, y = \infty$ in order to allow solutions corresponding to null fluid with positive mass and monopole spin (i.e. in which the integrated angular momentum is non-zero). As will appear in Section 6, these solutions require ψ, $A \sim \log(x^2 + y^2)$, so that one may add to both an arbitrary function of u. This destroys the uniqueness of ψ and A, but has no physical importance: because in the case of ψ only ψ_1 and ψ_2 occur in the g_{ik}, and in the case of A the additive function of u can be removed by a coordinate transformation of the type $x = x$, $y = y$, $v = v^* + \chi(u)$, $u = u$.

An example of a global solution is given in Section 6.

Finally, let us return to our discovery in Section 3 that the vacuum solution can be thrown into the form (3.5) by means of a coordinate transformation. This means that the exterior solution is always equivalent, at least locally, to a plane-fronted gravitational wave; moreover, it is known (Bonnor, 1969) that such waves can, if their sources are in the finite part of the x, y plane, be generated by null fluid without spin. The question therefore arises whether the global solutions of this section are isometric to global solutions for null fluid without spin.

This question may be answered negatively by considering the invariant λ discovered by Peres (1960).

$$R_{ik}R_{mn} = \lambda R^{a}_{ikc}R^{c}_{mna} \tag{4.3}$$

For space-times (2.1) which are empty but not flat, λ vanishes. If $T_{ik} \neq 0$, λ exists if $\alpha = \beta = 0$, but does not exist if w is a function of x and/or y. Hence there is a genuine difference between the interior solutions in the spinning and non-spinning cases, even though their exterior solutions are locally isometric. However, as already remarked, expression of the vacuum metric (2.1) in the form (3.5) may lead to topological difficulties. This is referred to again in Section 6.

5. *Physical Interpretation*

The energy tensor T_{ik} for the non-empty part of space-time is given by (2.16)–(2.18). T_{ik} has zero trace, and four zero eigenvalues. It has two linearly independent eigenvectors, one null and parallel to s^i given by (2.3), and one space-like, e.g.

$$X^i = (w_1, w_2, 0, 0)$$

It satisfies the equation

$$T_k{}^i T_i{}^m T_m{}^n = 0 \tag{5.1}$$

In the classification of Plebaski (1964) it is $[4N]_{(3)}$.

If $w_1 = w_2 = 0$ the energy tensor reduces to that of a beam of null fluid, and may be written

$$T_{ik} = \rho s_i s_k$$

s_i being obtained from (2.3) and ρ being the energy density (Bonnor, 1969). In this case, and in this case only, T_{ik} satisfies

$$T_k{}^i T_i{}^m = 0 \tag{5.2}$$

I shall therefore suppose that w represents some property superposed on a straight beam of null fluid.

Let us make the transformation

$$\sqrt{(2)}\,u = t - z, \qquad \sqrt{(2)}\,v = t + z \tag{5.3}$$

which takes (2.1) into

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 + A(dt - dz)^2 + \surd 2(\alpha\, dx + \beta\, dy)(dt - dz) \tag{5.4}$$

When the metric is used in this form I shall write

$$\bar{x}^1 \equiv x, \qquad \bar{x}^2 \equiv y, \qquad \bar{x}^3 \equiv z, \qquad \bar{x}^4 \equiv t \tag{5.5}$$

If α, β and A are small we can consider (5.4) as a perturbation of Minkowski space-time. Using (5.3) to transform the T^{ik} in (2.16)–(2.18) we find

$$\bar{T}_{33} = \bar{T}_{44} = -\bar{T}_{34} = -\bar{T}_{43} = \frac{1}{16\pi}(\nabla^2 A + \tfrac{1}{2}w^2) \tag{5.6}$$

$$\left.\begin{aligned} -\bar{T}_{13} = \bar{T}_{14} = \frac{\surd 2}{32\pi} w_2 \\ \bar{T}_{23} = -\bar{T}_{24} = \frac{\surd 2}{32\pi} w_1 \end{aligned}\right\} \tag{5.7}$$

Let us suppose for the moment that A and w are small, and interpret $-\bar{T}_{14}$ and $-\bar{T}_{24}$ as components of momentum density in the linear approximation theory. Integrating $\bar{T}_{14}$ over the cross-section of Σ [see (4.1)] by the two-surface $z = \text{const.}$, $t = \text{const.}$, we have

$$\int\int \bar{T}_{14}\, dx\, dy = \frac{\surd 2}{32\pi}\int dx \int \frac{\partial w}{\partial y}\, dy = 0$$

because w vanishes on the boundary of Σ, from (3.2). Similarly over this cross-section

$$\int\int \bar{T}_{24}\, dx\, dy = 0$$

Hence there is no total momentum in a two-surface $z = \text{const.}$, $t = \text{const.}$ Next consider

$$h(u) \stackrel{\text{def}}{=} \int\int (y\bar{T}_{14} - x\bar{T}_{24})\, dx\, dy = -\frac{\surd 2}{16\pi}\int\int w\, dx\, dy \tag{5.8}$$

h is the angular momentum about the z-axis per unit length of z-axis. *It follows that* $-\surd(2)\, w/16\pi$ *is the corresponding density of angular momentum.*

The above results refer to the linear approximation to our solutions. I shall take them as justification for interpreting the exact solutions as representing the interior and exterior fields of a stream of spinning null fluid moving in the z-direction. The stream is steady if w and A are independent of u.

In the case of the non-spinning null fluid, the energy density is given by (Bonnor, 1969a)

$$-\bar{T}_3{}^3 = -\bar{T}_3{}^4 = \bar{T}_4{}^3 = \bar{T}_4{}^4 \stackrel{\text{def}}{=} \rho \tag{5.9}$$

the bar referring to the coordinates of (5.5). Adopting the same definition of ρ in this case, we find

$$\rho = \frac{1}{32\pi}\left\{2\nabla^2 A + \frac{\partial}{\partial x}\left[\frac{\partial\psi}{\partial x}\nabla^2\psi\right] + \frac{\partial}{\partial y}\left[\frac{\partial\psi}{\partial y}\nabla^2\psi\right]\right\} \tag{5.10}$$

Integrating over a cross-section of Σ ,we obtain the mass per unit length

$$M(u) = \int\int \rho\, dx\, dy = \frac{1}{16\pi}\int\int \nabla^2 A\, dx\, dy \tag{5.11}$$

Two points of interest arise from the above. First, the angular momentum does not contribute to the total energy of the null fluid, although it enters the energy density as in (5.10). Secondly, consider a slice of null fluid inside Σ between the two hypersurfaces $u = k_1$, $u = k_2$ (k_1, k_2 constants). Since M is a function of u only, the mass of this slice is independent of the time. Because of (5.8) a similar result applies to the angular momentum. Thus the spinning null fluid does not radiate away mass or angular momentum. It is moving too fast: these quantities cannot escape from it.

The question arises whether the T_{ik} in (2.16)–(2.18) refers to a neutrino field. If so this T_{ik} should satisfy the Rainich equations for the neutrino field. A set of such equations has been given by Penney (1965), and my T_{ik} does not satisfy these. However, it seems that Penney's equations are somewhat restrictive. Indeed, Mr. J. B. Griffiths and Dr. R. A. Newing have kindly informed me that the T_{ik} in (2.16)–(2.18), together with the metric (2.1), can represent a neutrino field for certain choices of A and w. (See also Griffiths and Newing (1970)).

6. *Exact Solution for Spinning Null Fluid*

Consider the space-time given by (2.1) with

$r \stackrel{\text{def}}{=} +(x^2+y^2)^{1/2} > a$ (const.)

$$\left.\begin{aligned}
&\psi = -4\sqrt{(2)}\,\chi(u)\log r/a\\
&\alpha = -\frac{4\sqrt{(2)}\,\chi y}{r^2}, \qquad \beta = \frac{4\sqrt{(2)}\,\chi x}{r^2}\\
&A = 8m(u)\left(\log\frac{r}{a} + \frac{1}{2}\right)
\end{aligned}\right\} \tag{6.1}$$

$r < a$

$$\left.\begin{aligned}
&\psi = +\frac{2\sqrt{2}}{3}\chi\left[4\left(\frac{r}{a}\right)^3 - 9\left(\frac{r}{a}\right)^2 + 5\right]\\
&\alpha = \frac{4\sqrt{(2)}\,\chi y}{a^2}\left(2\frac{r}{a} - 3\right), \qquad \beta = -\frac{4\sqrt{(2)}\,\chi x}{a^2}\left(\frac{2r}{a} - 3\right)\\
&A = \frac{4mr^2}{a^2}
\end{aligned}\right\} \tag{6.2}$$

For $r > a$ we see that

$$w = \nabla^2 \psi = 0 \qquad \text{and} \qquad \nabla^2 A = 0 \tag{6.3}$$

so that the vacuum equations are satisfied in view of (3.1). Provided m and χ are suitable functions of u, the g_{ik} in (2.1) are now of class C^3 piecewise C^1, so we have a global solution of Einstein's equations. There is much arbitrariness about the interior solution ($r < a$): I have chosen it so that w and A are about the simplest possible algebraic functions of r.

For $r < a$ we find

$$w = \frac{24\sqrt{(2)}\chi}{a^2}\left(\frac{r}{a} - 1\right), \qquad h = \chi, \qquad M = m \tag{6.4}$$

(5.8) and (5.11) having been used. I therefore interpret the solution as referring to a stream of spinning null fluid, χ and m being the angular momentum and the mass, both per unit length. χ and m are arbitrary functions of u subject to differentiability. The energy density, given by (5.10), turns out to be

$$\rho = \frac{m}{\pi a^2} + \frac{6\chi^2}{\pi a^6}(8r^2 - 15ar + 6a^2) \tag{6.5}$$

The second term on the right-hand side has a minimum value of $-99\chi^2/16\pi a^4$ at $r = 15a/16$, hence if ρ is to be non-negative we need

$$m \geqslant 99\chi^2/16a^2 \tag{6.6}$$

By taking for $m(u)$ and $\chi(u)$ suitably smooth pulse functions we obtain a model for a particle of spinning null fluid, called a *spinning nullicon.* For example, we can take

$$m = k\chi = \begin{cases} 0, & |u| \geqslant b \\ (b^2 - u^2)^4, & |u| \leqslant b > 0 \end{cases} \tag{6.7}$$

where k is a constant chosen so that negative energies are avoided. The particle travels with the speed of light along the z axis of the coordinates of (5.5) carrying angular momentum and mass, both of which are constants of the motion. (In fact, of course, the profiles of the angular momentum density and energy density given by (6.7) propagate without change of shape.) The nullicon is accompanied by palne-fronted gravitational waves, as is shown below.

We turn now to the surprising feature of the exterior solution, mentioned in Section 3, namely, that α and β in (6.1) can be removed by a coordinate transformation, and the exterior metric reduced to the ordinary plane-fronted wave metric (3.5) The required transformation is

$$v = v^* - 4\sqrt{(2)}\chi(u)\tan^{-1}\left(\frac{y}{x}\right) \tag{6.8}$$

which takes (2.1) into

$$ds^2 = -dx^2 - dy^2 + 2\,dv^*\,du + 2\left(A - 4\sqrt{(2)}\,\chi' \tan^{-1}\frac{y}{x}\right) du^2 \quad (6.9)$$

where the prime means d/du. Let us assume that the sections $u = \text{const.}$ $v = \text{const.}$ have euclidean topology; then if $\chi' \neq 0$, either g_{44} is multivalued, or it is not continuous for all x, y. I therefore prefer to retain the metric in form (2.1) with α and β non-zero, and to rule that (6.8) is an inadmissible transformation.

The exceptional case, $\chi = \text{const.}$, refers to a steady straight flow of spinning null fluid (a steady beam of light with spin). In this case the angular momentum χ has no effect at all on the exterior metric. It seems remarkable that the spin in the source generates no Coriolis' forces in the surrounding empty space.

Something must be said about the conditions at $r = \infty$ of (6.1). To fix ideas let us take for $\chi(u)$ and $m(u)$ the function (6.7), so that we may think of the source as a moving particle. Then for $|u| \geqslant b$ the whole of (6.1) and (6.2) vanish and the space-time is flat. But for $|u| < b$ the fields are *stronger* than they would be for static spinning particle, which would have $\alpha, \beta \sim R^{-2}$ and $A \sim R^{-1}$, $R = (x^2 + y^2 + z^2)^{1/2}$. This is expected, since in electromagnetism and general relativity the wave field is stronger at great distance than the Coulomb field. Nevertheless, the appearance of the logarithm in A is somewhat unsatisfactory, because it means that, unless $m' = 0$, the Christoffel symbol Γ^3_{44} tends to infinity with r, which affects the equations of geodesics. On the other hand, one can set up natural (freely falling) coordinates round every point so that the space-time can be covered by non-singular coordinate neighbourhoods, albeit by an infinite number of them. The physical components of the Riemann tensor all tend to zero as $r \to \infty$.

7. *Superposition of Solutions*

Owing to the linearity of (2.11) and (3.2), vacuum solutions of metric (2.1) can be superposed. It is then easy to write down global solutions for several non-overlapping streams of spinning null fluid, each like (6.1)–(6.2). Thus, parallel beams of spinning null fluid do not interact, and in this respect resemble non-spinning null fluid.

8. *Conclusion*

The metric (2.1) is a generalisation of the space-time of a stream of null fluid, the latter being obtained if $\alpha = \beta = 0$. I have argued that the generalisation refers to spinning null fluid (SNF); my justification is the identification of the integral (5.8) with angular momentum. The integral refers of course to the interior field.

The spin is microscopic and does not refer to the rotation of null world-lines. To see this consider s^i, given by (2.3), which is tangent to a congruence of null geodesics for both interior and exterior solutions. For the interior solution one may consider particles of null fluid as moving along the congruence. However, since

$$s^i{}_{;k} = 0$$

the optical scalars (Kundt, 1961), including the rotation, all vanish. One has to think of the spin as a separate field superposed on the velocity field.

The interior solution for SNF is different from the interior solution for non-spinning null fluid (NSNF), because the invariant λ in (4.3) exists for the latter, but not for the former. (Also (5.2) is satisfied for the latter but not for the former.) However, an exterior metric for SNF is locally isometric to an exterior metric for NSNF. Nevertheless, except for the case of a steady beam of SNF, an exterior metric for SNF cannot be transformed globally into a regular NSNF metric with euclidean topology. This question is rather mysterious and needs further study.

References

Bonnor, W. B. (1969). *Communications in Mathematical Physics*, **13**, 163.
Bonnor, W. B. (1970). *International Journal of Theoretical Physics*, Vol. 3, No. 1, p. 57.
Ehlers, J. and Kundt, W. (1962). In: *Gravitation: An Introduction to Current Research*, p. 85, ed. Witten, L. John Wiley, New York.
Geroch, R. (1966). *Annals of Physics*, **36**, 147.
Griffiths, J. B. and Newing, R. A. (1970). *J. Phys. A: Gen. Phys.*, **3**, 269.
Kundt, W. (1961). *Zeitschrift für Physik*, **163**, 78.
Penney, R. (1965). *Journal of Mathematics and Physics*, **6**, 1309.
Peres, A. (1960). *Nuovo cimento*, **18**, 36.
Plebanski, J. (1964). *Acta physica polonica*, **26**, 963.
Wyman, M. and Trollope, R. (1965). *Journal of Mathematics and Physics*, **6**, 1995.